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# Replica method for wide correlators in Gaussian orthogonal, unitary and symplectic random matrix ensembles

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**Abstract.** We calculate connected correlators in Gaussian orthogonal, unitary and symplectic random matrix ensembles by the replica method in the $1/N$-expansion. We obtain averaged one-point Green's functions up to the next-to-leading order $O(1/N)$, wide two-level correlators up to the first non-trivial order $O(1/N^2)$ and wide three-level correlators up to the first non-trivial order $O(1/N^4)$ by carefully treating fluctuations in saddle-point evaluation.

## 1. Introduction

Recently, the universality of the wide-distance two-level connected correlator in random matrix theories was shown by Ambjørn *et al* [1]. Brézin and Zee [2] regarded this universality as important from the viewpoint of physics in disordered systems. Since the correlator does not depend on the probability distribution of a matrix ensemble, we may use the correlator calculated in the corresponding simple Gaussian ensemble with the same symmetry for the non-trivial disordered system in concern. Compared with the short-distance correlator which is already well known as a universal quantity [3], the wide correlators are able to be calculated explicitly in extensive types of ensembles in various ways [4, 5]. Actually, one observes the strongly universal properties which random matrix theories have themselves in the wide connected correlators. Their universality classification is also done in [6]. These mathematical studies of random matrix theories themselves enable us to recognize the real universal nature of level statistics.

In this paper, we examine the replica method to calculate the wide connected two-level correlator in Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles. These simple ensembles GOE, GUE and GSE can describe a time reversal system without spin, a general system and a time reversal system with spin, respectively [3]. The replica method is well known as a convenient scheme to calculate the two-body Green's function in some models of the Anderson localization [7]. Even though this method cannot calculate the short-distance correlator [8] well, here we show that this method is useful for calculating

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the wide two-level correlator in the $1/N$ expansion. This method is much simpler than the supersymmetry method which enables us to calculate both the wide- and short-distance correlators [8, 9]. Here, we remark the reason why the replica method is not good for the short-distance correlator. We are going to calculate the explicit form of the two- and three-level correlators in GOE, GUE and GSE, which are identical to those calculated by solving functional equations [4, 11] or by the diagrammatic method [12]. Here we study a matrix theory defined by the following probability distribution

$$P(H) = \frac{1}{Z_H} \exp\left(-\frac{N}{2} \operatorname{Tr} H^2\right) \qquad Z_H \equiv \int \mathrm{D}H \, \exp\left(-\frac{N}{2} \operatorname{Tr} H^2\right) \tag{1}$$

where $H$ is an $N \times N$ matrix. The explicit form of the measure $\mathrm{D}H$ depends on a type of an ensemble of $H$. Our interest is focused on computing averaged Green's functions by the replica method. We show their calculation method explicitly for GOE in section 2, for GUE in section 3 and GSE in section 4. We calculate three-level correlators in section 5.

## 2. Gaussian orthogonal ensemble

To begin with, we explore the case of the GOE which is defined by the ensemble of real symmetric matrices obeying the probability distribution (1). The measure $\mathrm{D}H$ is explicitly written as

$$\mathrm{D}H = \prod_{k=1}^{N} \mathrm{d}H_{kk} \prod_{i<j} \mathrm{d}H_{ij}. \tag{2}$$

### *2.1. One-point function*

We calculate the averaged one-point function

$$G(z) \equiv \left\langle \frac{1}{N} \operatorname{Tr} \frac{1}{z - H - \mathrm{i}\epsilon} \right\rangle \equiv \int \mathrm{D}H \, P(H) \frac{1}{N} \operatorname{Tr} \frac{1}{z - H - \mathrm{i}\epsilon}. \tag{3}$$

Note that $-\mathrm{i}\epsilon$ ($\epsilon > 0$) means that $G(z)$ considered here is the advanced Green's function. We assume

$$|z| > \sqrt{2} \tag{4}$$

in the case of the GOE. The meaning of this assumption will be clarified later. Results for $|z| < \sqrt{2}$ will be extracted employing the analytic continuation that is uniquely determined by $-\mathrm{i}\epsilon$.

In order to apply the replica method, we first introduce the $n$-flavour real vectors having $N$ component $(\phi_i^a)_{1 \leqslant i \leqslant N}$, where the superscript $a$, which runs from 1 to $n$, specifies the flavour. Let us consider the following integration over $\phi$ for constructing a generating function of $G(z)$:

$$\Gamma[z, H] \equiv \frac{1}{Z_\phi} \int \mathrm{D}\phi \, \exp\left\{-\frac{\mathrm{i}}{2} {}^t\phi^a (z - H - \mathrm{i}\epsilon)\phi^a\right\} \qquad Z_\phi = \int \mathrm{D}\phi \, \exp(-\tfrac{1}{2} {}^t\phi^a\phi^a) \tag{5}$$

where the superscript $t$ means transposition and the summation over the repeated indices are implicitly taken. The measure is defined by

$$\mathrm{D}\phi \equiv \prod_{a=1}^{n} \prod_{i=1}^{N} \mathrm{d}\phi_i^a. \tag{6}$$

Note that the integration over $\phi$ for $\Gamma$ converges owing to $-\mathrm{i}\epsilon$. Hereafter we do not explicitly write $-\mathrm{i}\epsilon$ for brevity.

Taking the derivative with respect to $z$, we have

$$\begin{aligned}\frac{\partial\Gamma[z,H]}{\partial z} &= \frac{\partial}{\partial z}\det{}^{-n/2}(\mathrm{i}(z-H)) \\ &= -\frac{n}{2}\operatorname{Tr}\frac{1}{z-H}\det{}^{-n/2}(\mathrm{i}(z-H)). \end{aligned} \tag{7}$$

We define $W(z)$ as the average of $\Gamma$ over $H$:

$$W(z) \equiv \langle\Gamma[z,H]\rangle. \tag{8}$$

It follows from equation (7) that

$$G(z) = \lim_{n\to 0}\left(-\frac{2}{nN}\right)\frac{\mathrm{d}}{\mathrm{d}z}W(z) \tag{9}$$

which indicates that $W(z)$ plays the role of a generating function of $G(z)$. Note that we need to take the limit $n \to 0$ in order to eliminate the determinant factor in equation (7).

Let us compute $W(z)$. The average over $H$ in equation (8) is explicitly carried out and we have

$$W(z) = \frac{1}{Z_\phi}\int \mathrm{D}\phi\ \exp\left(-\frac{\mathrm{i}}{2}\,{}^t\phi^a z\phi^a - \frac{1}{8N}({}^t\phi^a\phi^b)^2\right). \tag{10}$$

The quartic interaction can be represented as

$$\exp\left(-\frac{1}{8N}({}^t\phi^a\phi^b)^2\right) = \frac{1}{Z_Q}\int \mathrm{D}Q\ \exp\left(-\frac{N}{2}Q_{ab}Q_{ba} + \frac{\mathrm{i}}{2}\,{}^t\phi^a\phi^b Q_{ab}\right) \tag{11}$$

where $(Q)_{ab}$ is the real symmetric $n\times n$ matrix and the normalization factor is defined by

$$Z_Q \equiv \int \mathrm{D}Q\ \exp\left(-\frac{N}{2}Q_{ab}Q_{ba}\right). \tag{12}$$

Using representation (11) we obtain

$$W(z) = \frac{1}{Z_\phi Z_Q}\int \mathrm{D}Q\,\mathrm{D}\phi\ \exp\left(-\frac{N}{2}\operatorname{Tr}Q^2 - \frac{\mathrm{i}}{2}\,{}^t\phi_i(z-Q)\phi_i\right). \tag{13}$$

Note that indices $a$ and $b$ are implicitly summed in the above notation: we regard $\phi_i$ as an $n$-component vector.

The $\phi$-integral becomes Gaussian and we find

$$W(z) = \frac{1}{Z_Q}\int \mathrm{D}Q\ \exp\left\{-\frac{N}{2}\operatorname{Tr}(\log\mathrm{i}(z-Q) + Q^2)\right\}. \tag{14}$$

It can be written as integration over eigenvalues [3]

$$W(z) = \frac{C_n}{Z_Q}\int\prod_{a=1}^{n}\mathrm{d}u_a\,|\Delta(u)|\exp\left\{-N\sum_{a=1}^{n}g(u_a)\right\} \tag{15}$$

where $C_n$ is a constant that depends on $n$ and $\Delta(u)$ is the Van der Monde determinant

$$\Delta(u) \equiv \prod_{a<b}(u_a - u_b). \tag{16}$$

The function $g$ appeared in the exponent is defined by

$$g(x) \equiv \tfrac{1}{2}(\log\mathrm{i}(z-x) + x^2). \tag{17}$$

Now we compute $W(z)$ by the $1/N$-expansion. Let us first compute it in the leading order. Since the contribution from the Van der Monde determinant in equation (15) is neglected in the leading order, $u_a$'s completely decoupled each other and $W(z)$ is evaluated by a single-variable integration, i.e.

$$W(z) \sim \text{constant}\left(\int \mathrm{d}u\ \exp\{-Ng(u)\}\right)^n \tag{18}$$

in the leading order. We can evaluate the right-hand side of (18) by the saddle-point method. The saddle-point equation is

$$g'(u) = -\frac{1}{2(z-u)} + u = 0. \tag{19}$$

We have the two solutions $u_\pm$, where

$$u_\pm = \tfrac{1}{2}\left(z \pm \sqrt{z^2-2}\right). \tag{20}$$

We find that the contribution to integral (18) from $u \sim u_-$ dominates over that from $u \sim u_+$. In fact, a straightforward calculation gives

$$\operatorname{Re} g(u_-) < \operatorname{Re} g(u_+) \qquad \text{if } z > \sqrt{2}. \tag{21}$$

Therefore we obtain

$$W(z) \sim \text{constant } \mathrm{e}^{-Nng(u_-)} \tag{22}$$

in the leading order. Inserting the above result into equation (9), we obtain the well known result

$$G(z) = z - \sqrt{z^2-2} \tag{23}$$

where the overall constant in equation (22) is chosen such that $G(z)$ should satisfy the boundary condition $G(z) \to 1/z$ as $z \to \infty$.

We can compute higher-order corrections by expanding $g(u_a)$ in equation (15) around $u = u_-$

$$g(u_a) = g(u_-) + \frac{1}{2N} g''(u_-) y_a^2 + \sum_{n=3}^{\infty} \frac{N^{-n/2}}{n!} g^{(n)}(u_-) y_a^n \tag{24}$$

where we denote

$$u_a - u_- = N^{-1/2} y_a. \tag{25}$$

Here we shall calculate up to the first-order correction, which is sufficient for determining the connected two-point function, as we will see below. To this end, we can neglect the cubic and the higher terms in equation (24). Then we find

$$W(z) \sim C_n \mathrm{e}^{-Nng(u_-)} \int \prod_a \mathrm{d}y_a |\Delta(y)| \exp\left\{-\tfrac{1}{2}\sum_{a=1}^{n} g''(u_-) y_a^2\right\} \tag{26}$$

up to the next-to-leading order. Note that

$$\begin{aligned} g''(u_-) &= 1 - 2u_-^2 \\ &= \sqrt{z^2-2}\left(z - \sqrt{z^2-2}\right) \end{aligned} \tag{27}$$

which is real and positive under assumption (4). Therefore the integration over $y_a$ is easily performed going back to the representation of an integration over real symmetric matrices. Since there are $n(n+1)/2$ independent variables in a real symmetric matrix, we obtain

$$\frac{C_n}{Z_Q}\int\prod_a \mathrm{d}y_a\,|\Delta(y)|\exp\left\{-\frac{1}{2}\sum_{a=1}^n g''(u_-)y_a^2\right\} = \frac{1}{Z_Q}\int \mathrm{D}q\ \exp\left\{-\frac{1}{2}g''(u_-)\sum_{a,b}q_{ab}^2\right\}$$
$$= \left(\frac{1}{g''(u_-)}\right)^{n(n+1)/4}. \tag{28}$$

Thus we conclude

$$W(z)\sim \mathrm{e}^{-Nng(u_-)}\left(\frac{1}{g''(u_-)}\right)^{n(n+1)/4} \tag{29}$$

up to the next-to-leading order. Using equations (9) and (27), we obtain $G(z)$ including the first-order correction:

$$G(z) = \left(z-\sqrt{z^2-2}\right)\left(1+\frac{1}{2N}\frac{1}{z^2-2}\right)+\mathrm{O}(1/N^2). \tag{30}$$

This result is identical to that obtained in [11].

Here we comment on the validity of the saddle-point method in equation (28). The saddle-point evaluation in the $1/N$ expansion holds only in the region where $g''(u_-)$ is $\mathrm{O}(N^0)$. This approximation becomes incorrect for $z = \pm\sqrt{2}+\mathrm{O}(1/N)$ which yields $g''(u_-) = \mathrm{O}(1/N)$.

### 2.2. *Two-point function*

Now we turn to the two-point function

$$G(z_1,z_2)\equiv\left\langle\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{z_1-H}\,\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{z_2-H}\right\rangle. \tag{31}$$

We shall show how to compute *wide* correlation by the replica method, i.e. we assume $z_1 - z_2 \sim \mathrm{O}(N^0)$ as well as condition (4).

As is the case of the one-point function, we start with the following integral

$$\Gamma[z_1,z_2,H]\equiv\frac{1}{Z_{\phi_1}Z_{\phi_2}}\int \mathrm{D}\phi_1\,\mathrm{D}\phi_2\ \exp\left\{-\frac{\mathrm{i}}{2}\,{}^t\phi_1^a(z_1-H)\phi_1^a-\frac{\mathrm{i}}{2}\,{}^t\phi_2^p(z_2-H)\phi_2^p\right\}. \tag{32}$$

Here we have introduced the two species of vectors labelled by 1 and 2. The two species respectively have $n$- and $m$-flavour, which means that $1\leqslant a\leqslant n$ and $1\leqslant p\leqslant m$. We average $Z[z_1,z_2,H]$ over $H$ and define $W(z_1,z_2)$, a generating function of $G(z_1,z_2)$

$$W(z_1,z_2)\equiv\langle\Gamma[z_1,z_2,H]\rangle. \tag{33}$$

The two-point function is derived from $W(z_1,z_2)$ in the following formula:

$$G(z_1,z_2) = \lim_{n,m\to 0}\left(\frac{-2}{Nm}\right)\left(\frac{-2}{Nn}\right)\frac{\partial^2}{\partial z_1\partial z_2}W(z_1,z_2). \tag{34}$$

Our strategy for computing $W(z_1,z_2)$ is the same as in the case of the one-point function. Namely, we first perform the integration over $H$ of $W(z_1,z_2)$. Secondly, we introduce the auxiliary matrix $Q$ in order to carry out the integration over $\phi_1$ and $\phi_2$. The integration over the vector variables bring us the theory described by $Q$, which can be investigated by the saddle-point method for large $N$.

The first step is easily performed. We have

$$W(z_1, z_2) = \frac{1}{Z_{\phi_1} Z_{\phi_2}} \int \mathrm{D}\phi_1 \mathrm{D}\phi_2 \, \exp\left\{ -\frac{\mathrm{i}}{2}({}^t\phi_1^a z_1 \phi_1^a + {}^t\phi_2^p z_2 \phi_2^p) \right. \\ \left. -\frac{1}{8N}({}^t\phi_1^a\phi_1^b + {}^t\phi_2^p\phi_2^q)^2 \right\}. \tag{35}$$

Next, let us introduce the $(n+m) \times (n+m)$ real symmetric matrix $Q$ using the following expression:

$$Q = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \tag{36}$$

where $Q^{11}$ and $Q^{22}$ are $n \times n$ and $m \times m$ real symmetric matrices respectively. Since $Q$ is real symmetric, $Q^{12}$, which is an $n \times m$ matrix, satisfies $Q^{12} = {}^tQ^{21}$. In order to write down a formula corresponding to (13), we employ the following notations:

$$\Phi_i \equiv \begin{pmatrix} \phi_{1i} \\ \phi_{2i} \end{pmatrix} \tag{37}$$

which is $(n+m)$-component vectors, and

$$\boldsymbol{z} = \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix} \tag{38}$$

where $I_n$ is the unit $n \times n$ matrix. We can check that the right-hand side of equation (35) is identical to

$$\frac{1}{Z_Q Z_{\phi_1} Z_{\phi_2}} \int \mathrm{D}Q \, \mathrm{D}\phi_1 \, \mathrm{D}\phi_2 \, \exp\left\{ -\frac{N}{2} \operatorname{Tr} Q^2 - \frac{\mathrm{i}}{2} {}^t\Phi_i (\boldsymbol{z} - Q)\Phi_i \right\}. \tag{39}$$

After integrating the vector variables, we acquire

$$W(z_1, z_2) = \frac{1}{Z_Q} \int \mathrm{D}Q \, \exp\left\{ -\frac{N}{2} \operatorname{Tr}(\log \mathrm{i}(\boldsymbol{z} - Q) + Q^2) \right\}. \tag{40}$$

We analyse the above integral formula by the saddle-point method. The saddle-point equation is

$$(\boldsymbol{z} - Q)^{-1} = 2Q \tag{41}$$

or equivalently,

$$\begin{aligned} &2Q^{11}(z_1 - Q^{11}) - 2Q^{12}Q^{21} = 1 \\ &-2Q^{11}Q^{12} + 2Q^{12}(z_2 - Q^{22}) = 0 \\ &2Q^{21}(z_1 - Q^{11}) - 2Q^{22}Q^{21} = 0 \\ &-2Q^{21}Q^{12} + 2Q^{22}(z_2 - Q^{22}) = 1. \end{aligned} \tag{42}$$

After transposing the third equation and subtracting it from the second equation, we find

$$Q^{12} = Q^{21} = 0. \tag{43}$$

The remaining equations containing $Q^{11}$ and $Q^{22}$ are solved by diagonalization. Suppose that $Q^{11}$ and $Q^{22}$ are respectively diagonalized by $O_1$ and $O_2$. Explicitly,

$${}^tO_1 Q^{11} O_1 = \operatorname{diag}(u_1, \ldots, u_n) \qquad {}^tO_2 Q^{22} O_2 = \operatorname{diag}(v_1, \ldots, v_m). \tag{44}$$

Then the equations to be solved are

$$u_a = \frac{1}{2(z_1 - u_a)}$$
$$v_p = \frac{1}{2(z_2 - v_p)} \tag{45}$$

which have the same form as the saddle-point equation that appeared in section 2.1. Following the argument on the most dominant saddle point that is carried out in the one-point function, we should choose the saddle point

$$u_a = u_- \qquad \text{for all } a$$
$$v_p = v_- \qquad \text{for all } p. \tag{46}$$

Performing the inverse transformation of equation (44), we obtain the most dominant saddle point $\bar{Q}$ in the original basis:

$$\bar{Q} = \begin{pmatrix} u_- I_n & 0 \\ 0 & v_- I_m \end{pmatrix}. \tag{47}$$

Next we consider fluctuations around the saddle point $\bar{Q}$. Using the identity

$$\operatorname{Tr}\log(A + \delta A) = \operatorname{Tr}\log A + \operatorname{Tr} A^{-1}\delta A - \tfrac{1}{2}\operatorname{Tr} A^{-1}\delta A A^{-1}\delta A + \cdots \tag{48}$$

and the saddle-point equation (41), we have

$$\operatorname{Tr}\left(\log \mathrm{i}\left(z - \bar{Q} - \frac{\delta Q}{\sqrt{N}}\right) + \left(\bar{Q} + \frac{\delta Q}{\sqrt{N}}\right)^2\right)$$
$$= \operatorname{Tr}(\log \mathrm{i}(z - \bar{Q}) + \bar{Q}^2) + \frac{1}{N}\operatorname{Tr}(\delta Q^2 - 2\delta Q\bar{Q}\delta Q\bar{Q}) + \cdots. \tag{49}$$

Inserting the explicit form (47) into the above, we see that the generating function becomes

$$W(z_1, z_2) \sim \frac{1}{Z_Q}\mathrm{e}^{-Nng(u_-)}\int \mathrm{D}(\delta Q^{11})\ \exp\{-\tfrac{1}{2}(1 - 2u_-{}^2)\operatorname{Tr}(\delta Q^{11})^2\}$$
$$\times \mathrm{e}^{-Nmg(v_-)}\int \mathrm{D}(\delta Q^{22})\ \exp\{-\tfrac{1}{2}(1 - 2v_-{}^2)\operatorname{Tr}(\delta Q^{22})^2\}$$
$$\times \int \mathrm{D}(\delta Q^{12})\ \exp\{-(1 - 2u_-v_-)\operatorname{Tr}(\delta Q^{12})^2\} \tag{50}$$

up to the next-to-leading order. Remembering that $g''(u_-) = 1 - 2u_-^2$, we obtain

$$W(z_1, z_2) = \mathrm{e}^{-Nng(u_-)}\left(\frac{1}{g''(u_-)}\right)^{n(n+1)/4}\mathrm{e}^{-Nmg(v_-)}\left(\frac{1}{g''(v_-)}\right)^{m(m+1)/4}\left(\frac{1}{1 - 2u_-v_-}\right)^{nm/2}$$
$$= W(z_1)W(z_2)\left(\frac{1}{1 - 2u_-v_-}\right)^{nm/2}. \tag{51}$$

Inserting the above formula into equation (34), we can derive the two-point function. Differentiating $W(z_1)$ and $W(z_2)$ brings about the disconnected part, $G(z_1)G(z_2)$, while the last factor contributes to the connected part. Thus we obtain

$$G(z_1, z_2) = G(z_1)G(z_2) - \frac{2}{N^2}\frac{\partial^2}{\partial z_1 \partial z_2}\log\left(1 - \frac{1}{2}G(z_1)G(z_2)\right) \tag{52}$$

where we have used equation (9) and $2u_- = G(z_1) + \mathrm{O}(1/N)$. This result, with respect to the connected part, agrees with that obtained by solving functional equations [4, 11] and by a diagrammatic method [12]. Especially the disconnected part up to $1/N^2$ order in equation (52) is consistent with [11].

The saddle-point evaluation holds only in the case of $1 - 2u_- v_- = \mathrm{O}(N^0)$, as pointed out in the calculation of the one-point Green's function. When we compute the following two-level correlators:

$$\left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z_1 - H + \mathrm{i}\epsilon} \frac{1}{N} \mathrm{Tr} \frac{1}{z_2 - H - \mathrm{i}\epsilon} \right\rangle \tag{53}$$

the coefficient of Gaussian fluctuations becomes $1 - 2u_+ v_-$. The result of the saddle-point method should be trusted only for $z_1 - z_2 = \mathrm{O}(N^0)$ indicated by the definition of $u_\pm$ equation (20). Therefore, the short-distance two-level correlators should be calculated using other methods instead of the saddle-point evaluation.

## 3. Gaussian unitary ensemble

We turn to the case of the GUE, which means the ensemble of Hermitian matrices following the Gaussian distribution (1). The measure $\mathrm{D}H$ is explicitly written as

$$\mathrm{D}H \equiv \prod_{i \leqslant j} \mathrm{d}(\mathrm{Re}\, H_{ij}) \prod_{i<j} \mathrm{d}(\mathrm{Im}\, H_{ij}). \tag{54}$$

We can proceed with a computation as in the case of GOE. We assume

$$|z| > 2 \tag{55}$$

in the case of the GUE, which corresponds to assumption (4). We can construct a generating function $W(z)$ by employing *complex* vector variables as follows

$$W(z) \equiv \left\langle \frac{1}{Z_\phi} \int \mathrm{D}\phi \ \exp\left\{ -\frac{\mathrm{i}}{2} \phi^{a\dagger} (z - H) \phi^a \right\} \right\rangle \tag{56}$$

where

$$\mathrm{D}\phi \equiv \prod_{a=1}^{n} \prod_{i=1}^{N} \mathrm{d}(\mathrm{Re}\, \phi_i^a)\, \mathrm{d}(\mathrm{Im}\, \phi_i^a). \tag{57}$$

We can readily show that

$$G(z) = \lim_{n \to 0} \left( -\frac{1}{nN} \right) \frac{\mathrm{d}}{\mathrm{d}z} W(z) \tag{58}$$

in a similar way to the derivation of (9). A difference between equation (9) and equation (58) arises because the complex vector variables contribute twice compared with the case of real vector variables. We first take the average over $H$ in (56) and we introduce a Hermitian auxiliary matrix $Q_{ab}$ along the line with the case of the GOE. See equations (10)–(13). Then the integration over the complex vector variables gives

$$\begin{aligned} W(z) &= \frac{1}{Z_Q} \int \mathrm{D}Q \ \exp\left\{ -\frac{N}{2} \mathrm{Tr}(2 \log \mathrm{i}(z - Q) + Q^2) \right\} \\ &= \frac{C_n}{Z_Q} \int \prod_{a=1}^{n} \mathrm{d}w_a\, \Delta(w)^2 \exp\left\{ -\frac{N}{2} \sum_{a=1}^{n} (2 \log \mathrm{i}(z - w_a) + {w_a}^2) \right\}. \end{aligned} \tag{59}$$

Diagonalizing $Q$, we obtain the saddle-point equation corresponding to equation (19)

$$-\frac{1}{z - w} + w = 0. \tag{60}$$

The most dominant saddle point is $w_- I_n$, in which

$$w_- \equiv \tfrac{1}{2}(z - \sqrt{z^2 - 4}). \tag{61}$$

We expand the exponent of (59) around $w_- I_n$ and ignore the cubic and higher terms of the fluctuation in the same way as we did for GOE. The Gaussian integration of the fluctuation is easily performed, from which we derive

$$W(z) \sim \exp\left\{-\frac{Nn}{2}(2\log(z - w_-) + w_-^2)\right\}\left(\frac{1}{1 - w_-^2}\right)^{n^2/2} \tag{62}$$

up to the next-to-leading order. Using this result and equation (58), we obtain

$$G(z) = \tfrac{1}{2}\left(z - \sqrt{z^2 - 4}\right) + \mathrm{O}(1/N^2). \tag{63}$$

Note that first-order corrections vanish in this case.

The two-point function defined in equation (31) is also calculated following the case of GOE. Using complex vectors, we define the generating function $W(z_1, z_2)$ as

$$W(z_1, z_2) \equiv \left\langle \frac{1}{Z_{\phi_1} Z_{\phi_2}} \int \mathrm{D}\phi_1\, \mathrm{D}\phi_2\, \exp\left\{-\frac{\mathrm{i}}{2}\phi_1^{a\dagger}(z_1 - H)\phi_1^a - \frac{\mathrm{i}}{2}\phi_2^{p\dagger}(z_2 - H)\phi_2^p\right\}\right\rangle. \tag{64}$$

The two-point function $G(z_1, z_2)$ is derived from $W(z_1, z_2)$ as

$$G(z_1, z_2) = \lim_{n,m\to 0}\left(\frac{-1}{Nm}\right)\left(\frac{-1}{Nn}\right)\frac{\partial^2}{\partial z_1 \partial z_2} W(z_1, z_2). \tag{65}$$

We can write $W(z_1, z_2)$ in terms of an integration over Hermitian $(n+m)\times(n+m)$ matrices, which corresponds to equation (40) for the GOE case. The result is

$$W(z_1, z_2) = \frac{1}{Z_Q}\int \mathrm{D}Q\, \exp\left\{-\frac{N}{2}\,\mathrm{Tr}(2\log \mathrm{i}(\boldsymbol{z} - Q) + Q^2)\right\}. \tag{66}$$

The most dominant saddle point $\bar{Q}$ is

$$\bar{Q} = \begin{pmatrix} w_1 I_n & 0 \\ 0 & w_2 I_m \end{pmatrix} \tag{67}$$

where

$$w_i \equiv \tfrac{1}{2}\left(z_i - \sqrt{z_i{}^2 - 4}\right) \qquad i = 1, 2. \tag{68}$$

Since the connected part of the two-point function is the order $\mathrm{O}(1/N^2)$, we must take in the effect of fluctuations around $\bar{Q}$. As we explained in the GOE case, we can regard the fluctuations as Gaussian for the leading order of the connected part. Following the step to derive equation (51), we obtain

$$W(z_1, z_2) = W(z_1)W(z_2)\left(\frac{1}{1 - w_1 w_2}\right)^{nm}. \tag{69}$$

Employing equation (65), the above result is translated to the language of the Green's function:

$$G(z_1, z_2) = G(z_1)G(z_2) - \frac{1}{N^2}\frac{\partial^2}{\partial z_1 \partial z_2}\log(1 - G(z_1)G(z_2)). \tag{70}$$

This agrees with the well known result obtained by several other methods [1, 2, 4, 11, 12].

## 4. Gaussian symplectic ensemble

### 4.1. Definition

The GSE is the ensemble of quaternion real Hermitian matrices with the distribution (1). We shall first recall the definition of a quaternion real Hermitian matrix.

Let us write

$$H = \begin{pmatrix} H_{11} & \dots & H_{1N} \\ H_{21} & \dots & H_{2N} \\ \dots\dots & \dots & \dots\dots \\ H_{N1} & \dots & H_{NN} \end{pmatrix} \tag{71}$$

where each entry $H_{ij}$ is a quaternion number, which can be represented as a $2 \times 2$ matrix. We choose the set of the Pauli matrices $\sigma^k$ and the unit matrix $I_2$ as a basis of $2\times2$ matrices. Then $H_{ij}$ is written as

$$\begin{aligned} H_{ij} &= H_{ij}^{(0)} I_2 + \mathrm{i} \sum_{k=1}^{3} H_{ij}^{(k)} \sigma^k \\ &\equiv \begin{pmatrix} H_{ij}^{11} & H_{ij}^{12} \\ H_{ij}^{21} & H_{ij}^{22} \end{pmatrix}. \end{aligned} \tag{72}$$

In our notation, $i, j = 1, \dots, N$ and $\alpha, \beta = 1, 2$ in $H_{ij}^{\alpha\beta}$. The matrix $H$ is called a quaternion real Hermitian matrix if and only if $H_{ij}^{(0)}$ forms a real symmetric matrix whereas $H_{ij}^{(k)}$ $(k = 1, 2, 3)$ form real antisymmetric matrices. These conditions are alternatively expressed as

$$\begin{aligned} &H_{ij}^{11*} = H_{ij}^{22} \\ &H_{ij}^{12*} = -H_{ij}^{21} \\ &H_{ij}^{\dagger} = H_{ji}. \end{aligned} \tag{73}$$

Note that $^\dagger$ in the last equation means the Hermitian conjugation when we regard $H_{ij}$ as a $2 \times 2$ matrix.

The trace of $H^2$ in equation (1) reads

$$\mathrm{Tr}\, H^2 = H_{ij}^{\alpha\beta} H_{ji}^{\beta\alpha} \tag{74}$$

where $\alpha, \beta = 1, 2$. Using $H_{ij}^{(\mu)}$, $\mu = 1, \dots 4$, the measure D$H$ in this case is written as

$$\mathrm{D}H = \prod_{i \leqslant j} \mathrm{d}H_{ij}^{(0)} \prod_{k=1}^{3} \prod_{i<j} \mathrm{d}H_{ij}^{(k)}. \tag{75}$$

Here, for later convenience, we consider how to construct a quaternion real Hermitian matrix from a complex vector with an even-number component. Let $\phi_i^{a\alpha}$ $(1 \leqslant i \leqslant N, 1 \leqslant a \leqslant n, \alpha = 1, 2)$ be a complex number. We regard $\phi^a$'s as the $2n$-component vectors. The diadig

$$\tilde{\kappa}(\phi^a) \equiv \phi^a \phi^{a\dagger} \qquad \text{or} \qquad \tilde{\kappa}_{ji}^{\alpha\beta} \equiv \phi_j^{a\beta} \phi_i^{a\alpha *} \tag{76}$$

defines the $2N \times 2N$ Hermitian matrix. For constructing a quaternion real Hermitian matrix, we need to 'symmetrize' $\tilde{\kappa}(\phi^a)$ in the following way:

$$\begin{aligned}
\tilde{\kappa}_{ji}^{11} &\to \tfrac{1}{2}(\tilde{\kappa}_{ji}^{11} + \tilde{\kappa}_{ij}^{22}) \equiv \kappa_{ji}^{11} \\
\tilde{\kappa}_{ji}^{12} &\to \tfrac{1}{2}(\tilde{\kappa}_{ji}^{12} - \tilde{\kappa}_{ij}^{12}) \equiv \kappa_{ji}^{12} \\
\tilde{\kappa}_{ji}^{21} &\to \tfrac{1}{2}(\tilde{\kappa}_{ji}^{21} - \tilde{\kappa}_{ij}^{21}) \equiv \kappa_{ji}^{21} \\
\tilde{\kappa}_{ji}^{22} &\to \tfrac{1}{2}(\tilde{\kappa}_{ji}^{22} + \tilde{\kappa}_{ij}^{11}) \equiv \kappa_{ji}^{22}.
\end{aligned} \tag{77}$$

The matrix $\kappa(\phi^a)$ defined above is a quaternion real Hermitian matrix since it satisfies the condition

$$\begin{aligned}
\kappa_{ij}^{11*} &= \kappa_{ij}^{22} \\
\kappa_{ij}^{12*} &= -\kappa_{ij}^{21} \\
\kappa_{ij}^{\dagger} &= \kappa_{ji}.
\end{aligned} \tag{78}$$

Another useful quaternion real Hermitian matrix is constructed as follows. We define

$$\psi_i{}^{a1} \equiv \phi_i{}^{a1} \qquad \psi_i{}^{a2} \equiv \phi_i{}^{a2*} \tag{79}$$

and make the diadig $\psi_i \psi_i^{\dagger}$. By the same symmetrization as in equation (77), we obtain the $2n \times 2n$ quaternion real Hermitian matrix. We shall denote the resultant matrix by $\lambda(\phi_i)$. A straightforward calculation gives

$$\mathrm{Tr}\,\kappa(\phi^a)^2 = \mathrm{Tr}\,\lambda(\phi_i)^2. \tag{80}$$

### *4.2. One-point function*

Let us compute the one-point function $G(z)$ averaged over GSE

$$G(z) \equiv \left\langle \frac{1}{2N}\,\mathrm{Tr}\,\frac{1}{z-H} \right\rangle. \tag{81}$$

Since $H$ is a $2N \times 2N$ matrix, $G(z)$ tends to $1/z$ as $z \to \infty$. We assume (55) as in the case of the GUE. The generating function $W(z)$ is defined by

$$W(z) \equiv \left\langle \frac{1}{Z_\phi} \int \mathrm{D}\phi\, \exp\left\{ -\frac{\mathrm{i}}{2} \phi^{a\dagger}(z-H)\phi^a \right\} \right\rangle. \tag{82}$$

Note that $\phi^a$ is a $2N$-component vector, which implies

$$\phi^{a\dagger} H \phi^a \equiv \phi_i^{a\alpha*} H_{ij}^{\alpha\beta} \phi_j^{a\beta} \tag{83}$$

in equation (82). The one-point function is derived from $W(z)$ as

$$G(z) = \lim_{n\to 0} \left( -\frac{1}{2nN} \right) \frac{\mathrm{d}}{\mathrm{d}z} W(z). \tag{84}$$

Let us take the average over $H$ in equation (82). We first note that

$$\phi^{a\dagger} H \phi^a = \mathrm{Tr}\,\tilde{\kappa}(\phi^a) H. \tag{85}$$

We can symmetrize $\tilde{\kappa}(\phi^a)$ using the following relations

$$H_{ij}^{11} = H_{ji}^{22} \qquad H_{ij}^{12} = -H_{ji}^{12} \tag{86}$$

which are derived from equations (73). For example, the coupling with the diagonal part of $H_{ij}$ in equation (85) becomes

$$\begin{aligned}
\tilde{\kappa}_{ji}^{11} H_{ij}^{11} + \tilde{\kappa}_{ji}^{22} H_{ij}^{22} &= \tfrac{1}{2}(\tilde{\kappa}_{ji}^{11} + \tilde{\kappa}_{ij}^{22}) H_{ij}^{11} + \tfrac{1}{2}(\tilde{\kappa}_{ij}^{11} + \tilde{\kappa}_{ji}^{22}) H_{ij}^{22} \\
&= \kappa_{ji}^{11} H_{ij}^{11} + \kappa_{ji}^{22} H_{ij}^{22}.
\end{aligned} \tag{87}$$

After a similar procedure is performed on $H_{ij}^{12}$ and $H_{ij}^{21}$, it turns out that the matrix $\kappa(\phi^a)$ defined in equation (77) can replace $\tilde{\kappa}(\phi^a)$ in equation (85).

We are now ready to take the average over $H$ in equation (82). That is,

$$-\frac{N}{2}\,\mathrm{Tr}\,H^2 + \frac{\mathrm{i}}{2}\phi^{a\dagger}H\phi^a = -\frac{N}{2}\,\mathrm{Tr}\left(H - \frac{\mathrm{i}}{2N}\kappa(\phi^a)\right)^2 - \frac{1}{8N}\,\mathrm{Tr}\,\kappa(\phi^a)^2. \tag{88}$$

The change $H \to H + \frac{\mathrm{i}}{2N}\kappa(\phi^a)$ can be performed by shifting the integration variables $H_{ij}^{(\mu)}$ because $\kappa(\phi^a)$ is a quaternion real Hermitian matrix. It leads to

$$W(z) = \frac{1}{Z_\phi}\int \mathrm{D}\phi\, \exp\left(-\frac{\mathrm{i}}{2}\phi^{a\dagger}z\phi^a - \frac{1}{8N}\,\mathrm{Tr}\,\kappa(\phi^a)^2\right). \tag{89}$$

Next, for the sake of integration over $\phi$, we introduce the auxiliary matrix $Q$ that is a $2n \times 2n$ quaternion real Hermitian matrix. Let us consider the following integration

$$\frac{1}{Z_Q}\int \mathrm{D}Q\, \exp\left(-\frac{N}{2}Q_{ab}^{\alpha\beta}Q_{ba}^{\beta\alpha} + \frac{\mathrm{i}}{2}\psi_i^{a\alpha *}Q_{ab}^{\alpha\beta}\psi_i^{b\beta}\right) \tag{90}$$

where $\psi_i$'s $(i = 1, \ldots n)$ are defined in equation (79).

The $Q$-integral in equation (90) is carried out in the same way as the $H$-integral of equation (82). The result is

$$\frac{1}{Z_Q}\int \mathrm{D}Q\, \exp\left(-\frac{N}{2}Q_{ab}^{\alpha\beta}Q_{ba}^{\beta\alpha} + \frac{\mathrm{i}}{2}\psi_i^{a\alpha *}Q_{ab}^{\alpha\beta}\psi_i^{b\beta}\right) = \exp\left(-\frac{1}{8N}\,\mathrm{Tr}\,\lambda(\phi^a)^2\right). \tag{91}$$

Thus, from equations (79), (80), (89) and (91), we conclude

$$W(z) = \frac{1}{Z_\phi Z_Q}\int \mathrm{D}Q\,\mathrm{D}\phi\,\left(-\frac{N}{2}\,\mathrm{Tr}\,Q^2 - \frac{\mathrm{i}}{2}{\psi_i}^\dagger(z - Q)\psi_i\right). \tag{92}$$

Performing integration over $\phi$, we obtain

$$W(z) = \frac{1}{Z_Q}\int \mathrm{D}Q\, \exp\left\{-\frac{N}{2}\,\mathrm{Tr}(2\log\mathrm{i}(z - Q) + Q^2)\right\}. \tag{93}$$

Since any eigenvalue of $Q$ has two-fold degeneracy [3], the eigenvalue representation of the above integration becomes

$$W(z) = \frac{1}{Z_Q}\int \prod_{a=1}^{n}\mathrm{d}w_a\,\Delta(w)^4 \exp\left\{-N\sum_{a=1}^{n}(2\log\mathrm{i}(z - w_a) + {w_a}^2)\right\}. \tag{94}$$

Repeating the same argument as in the case of GUE, we obtain the result up to the next-to leading order:

$$W(z) \sim \exp\{-Nn(2\log(z - w_-) + w_-^2)\}\left(\frac{1}{1 - w_-^2}\right)^{n^2 - n/2} \tag{95}$$

where $w_-$ is defined in equation (61). The exponent of $(1 - w_-^2)$ is different from the result of GUE because the degree of freedom of fluctuations in the case of the GUE is given by $n^2$ while $2n^2 - n$ in the case of the GSE. That affects the non-vanishing first-order correction to $G(z)$. Namely, from equation (84), we obtain

$$G(z) = \frac{1}{2}(z - \sqrt{z^2 - 4})\left(1 - \frac{1}{2N}\frac{1}{z^2 - 4}\right) + \mathrm{O}(1/N^2). \tag{96}$$

*4.3. Two-point function*

Next we compute the two-point function $G(z_1, z_2)$ defined by

$$G(z_1, z_2) \equiv \left\langle \frac{1}{2N} \operatorname{Tr} \frac{1}{z_1 - H} \frac{1}{2N} \operatorname{Tr} \frac{1}{z_2 - H} \right\rangle. \tag{97}$$

The generating function $W(z_1, z_2)$ has the same form as the GUE case:

$$W(z_1, z_2) \equiv \left\langle \frac{1}{Z_{\phi_1} Z_{\phi_2}} \int \mathrm{D}\phi_1 \, \mathrm{D}\phi_2 \, \exp\left\{ -\frac{\mathrm{i}}{2} \phi_1^{a\dagger} (z_1 - H) \phi_1^a - \frac{\mathrm{i}}{2} \phi_2^{p\dagger} (z_2 - H) \phi_2^p \right\} \right\rangle. \tag{98}$$

The two-point function $G(z_1, z_2)$ is related to $W(z_1, z_2)$ as follows

$$G(z_1, z_2) = \lim_{n,m \to 0} \left( \frac{-1}{2Nm} \right) \left( \frac{-1}{2Nn} \right) \frac{\partial^2}{\partial z_1 \partial z_2} W(z_1, z_2). \tag{99}$$

In order to perform the integration over $H$, we introduce the notation $\Phi_i^{A\alpha}$ as

$$\Phi_i^{A\alpha} \equiv \begin{cases} \phi_{1i}^{A\alpha} & 1 \leqslant A \leqslant n \\ \phi_{2i}^{A-n\alpha} & n+1 \leqslant A \leqslant n+m. \end{cases} \tag{100}$$

Then, since the coupling to $H$ can be rewritten as $\Phi^{A\dagger} H \Phi^A$, we obtain the following result as the integration over $H$:

$$W(z_1, z_2) = \frac{1}{Z_{\phi_1} Z_{\phi_2}} \int \mathrm{D}\phi_1 \, \mathrm{D}\phi_2 \, \exp\left\{ -\frac{\mathrm{i}}{2} \Phi^{A\dagger} \boldsymbol{z} \Phi^A - \frac{1}{8N} \operatorname{Tr} \kappa (\Phi^A)^2 \right\} \tag{101}$$

where $\boldsymbol{z}$ is the following $2(n+m) \times 2(n+m)$ matrix

$$\boldsymbol{z} = \begin{pmatrix} z_1 I_{2n} & 0 \\ 0 & z_1 I_{2m} \end{pmatrix}. \tag{102}$$

According to formulae (80) and (91), we can express $W(z_1 z_2)$ by $2(n+m) \times 2(n+m)$ matrix $Q$:

$$\begin{aligned} W(z_1, z_2) &= \frac{1}{Z_{\phi_1} Z_{\phi_2} Z_Q} \int \mathrm{D}\phi_1 \, \mathrm{D}\phi_2 \, \mathrm{D}Q \, \exp\left\{ -\frac{\mathrm{i}}{2} \Phi_i{}^\dagger \boldsymbol{z} \Phi_i - \frac{N}{2} \operatorname{Tr} Q^2 + \frac{\mathrm{i}}{2} \Psi_i{}^\dagger Q \Psi_i \right\} \\ &= \frac{1}{Z_{\phi_1} Z_{\phi_2} Z_Q} \int \mathrm{D}\phi_1 \, \mathrm{D}\phi_2 \, \mathrm{D}Q \, \exp\left\{ -\frac{N}{2} \operatorname{Tr} Q^2 - \frac{\mathrm{i}}{2} \Psi_i{}^\dagger (z - Q) \Psi_i \right\} \end{aligned} \tag{103}$$

where $\Psi$ is defined in the same way as equation (79):

$$\Psi_i{}^{A1} \equiv \phi_i{}^{A1} \qquad \Psi_i{}^{A2} \equiv \phi_i{}^{A2*}. \tag{104}$$

After the integration over the vector variables, we obtain

$$W(z_1, z_2) = \frac{1}{Z_Q} \int \mathrm{D}Q \, \exp\left\{ -\frac{N}{2} \operatorname{Tr}(2 \log \mathrm{i}(\boldsymbol{z} - Q) + Q^2) \right\}. \tag{105}$$

The most dominant saddle point $\bar{Q}$ is

$$\bar{Q} = \begin{pmatrix} w_1 I_{2n} & 0 \\ 0 & w_2 I_{2m} \end{pmatrix}. \tag{106}$$

Next we compute effects of fluctuations around $\bar{Q}$

$$W(z_1, z_2) = W(z_1) W(z_2) \left( \frac{1}{1 - w_1 w_2} \right)^{2nm}. \tag{107}$$

Employing equation (65), the above result is translated to the language of the Green's function:

$$G(z_1, z_2) = G(z_1)G(z_2) - \frac{1}{2N^2}\frac{\partial^2}{\partial z_1 \partial z_2}\log(1 - G(z_1)G(z_2)). \tag{108}$$

This result, with respect to the connected part, agrees with that obtained by solving functional equations [4, 11] and by a diagrammatic method [12]. Especially the disconnected part up to $1/N^2$ order in equation (108) is consistent with [11].

## 5. Three-point function

Finally, we calculate the connected three-point function in each Gaussian ensemble. That in the GOE case is defined by

$$G(z_1, z_2, z_3) \equiv \left\langle \frac{1}{N}\,\mathrm{Tr}\,\frac{1}{z_1 - H}\,\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{z_2 - H}\,\frac{1}{N}\,\mathrm{Tr}\,\frac{1}{z_3 - H}\right\rangle. \tag{109}$$

As is the case of one- and two-point functions, we construct the generating function of $G(z_1, z_2, z_3)$ using three $n$-flavour *real* vectors:

$$W(z_1, z_2, z_3) = \left\langle \frac{1}{Z_{\phi_1} Z_{\phi_2} Z_{\phi_3}} \int \prod_{i=1}^{3} \mathrm{D}\phi_i\, \exp\left\{ - \sum_{j=1}^{3} \frac{\mathrm{i}}{2}\, {}^t\phi_j^a (z_j - H)\phi_j^a \right\}\right\rangle. \tag{110}$$

The three-point function $G(z_1, z_2, z_3)$ is derived from $W(z_1, z_2, z_3)$ as

$$G(z_1, z_2, z_3) = \lim_{n\to 0}\left(\frac{-2}{Nn}\right)^3 \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} W(z_1, z_2, z_3). \tag{111}$$

We can write $W(z_1, z_2, z_3)$ in terms of an integration over $3n \times 3n$ real symmetric matrix. The result is

$$W(z_1, z_2, z_3) = \frac{1}{Z_Q}\int \mathrm{D}Q\, \exp\left\{ -\frac{N}{2}\,\mathrm{Tr}(\log \mathrm{i}(\boldsymbol{z} - Q) + Q^2)\right\}. \tag{112}$$

The most dominant saddle point $\bar{Q}$ is

$$\bar{Q} = \begin{pmatrix} w_1 I_n & 0 & 0 \\ 0 & w_2 I_n & 0 \\ 0 & 0 & w_3 I_n \end{pmatrix} \tag{113}$$

where

$$w_i \equiv \tfrac{1}{2}\left(z_i - \sqrt{{z_i}^2 - 2}\right) \qquad i = 1, 2, 3. \tag{114}$$

Next we consider fluctuations around the saddle point $\bar{Q}$.

$$W(z_1, z_2, z_3) = \frac{\mathrm{e}^{-S_0}}{Z_Q}\int \mathrm{D}(\delta Q)\, \mathrm{e}^{-S_2 - \sum_{l=3}^{\infty} S_l} \tag{115}$$

where

$$S_0 = \frac{N}{2}\,\mathrm{Tr}(\log \mathrm{i}(\boldsymbol{z} - \bar{Q}) + \bar{Q}^2) \tag{116}$$

$$\begin{aligned} S_2 &= \tfrac{1}{2}\,\mathrm{Tr}((\delta Q)^2 - (\bar{Q}\delta Q)^2) \\ &= \tfrac{1}{2}\sum_{i,j=1}^{3}\sum_{a,b=1}^{n}(1 - w_i w_j)(\delta Q_{ab}^{ij})^2 \end{aligned} \tag{117}$$

$$S_l = \frac{1}{l N^{l/2-1}}\,\mathrm{Tr}(\bar{Q}\delta Q)^l \qquad l > 3. \tag{118}$$

Since the connected part of three-point function is of the order $1/N^4$, we must calculate the following Gaussian integral with $S_4$ and $S_3^2$ terms:

$$W(z_1, z_2, z_3) = \frac{\mathrm{e}^{-S_0}}{Z_Q} \int \mathrm{D}(\delta Q)\, \mathrm{e}^{-S_2} \left\{ 1 - \sum_{l=3}^{\infty} S_l + \frac{1}{2!} \left( \sum_{l=3}^{\infty} S_l \right)^2 - \cdots \right\}$$
$$= \mathrm{e}^{-S_0} \left\{ 1 - \langle\!\langle S_4 \rangle\!\rangle + \frac{1}{2!} \langle\!\langle S_3^2 \rangle\!\rangle + \mathrm{O}(1/N^5) \right\} \tag{119}$$

where

$$\langle\!\langle \cdots \rangle\!\rangle = \frac{1}{Z_Q} \int \mathrm{D}(\delta Q)(\cdots)\, \mathrm{e}^{-S_2}. \tag{120}$$

Calculating equation (119), we have

$$W(z_1, z_2, z_3) = \frac{n^3}{2N} F(z_1, z_2, z_3) \tag{121}$$

where

$$F(z_1, z_2, z_3) = \frac{X_{12}}{1 - X_{12}} \frac{X_{23}}{1 - X_{23}} \frac{X_{31}}{1 - X_{31}} + \frac{X_{31}}{1 - X_{31}} \frac{X_{12}}{1 - X_{12}} \frac{1}{1 - X_{11}}$$
$$+ \frac{X_{12}}{1 - X_{12}} \frac{X_{23}}{1 - X_{23}} \frac{1}{1 - X_{22}} + \frac{X_{23}}{1 - X_{23}} \frac{X_{31}}{1 - X_{31}} \frac{1}{1 - X_{33}} \tag{122}$$

$$X_{ij} = 2 w_i w_j = \tfrac{1}{2} G(z_i) G(z_j). \tag{123}$$

Substituting the above formula into equation (111), we can derive the three-point function

$$G_C(z_1, z_2, z_3) = -\frac{4}{N^4} \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} F(z_1, z_2, z_3). \tag{124}$$

The connected part in this result agrees with that obtained by other methods [10–12].

Next we turn to the connected three-point function in the GUE case along the way in the case of GOE. We construct a generating function $W(z_1, z_2, z_3)$ by employing three *complex* vectors as follows

$$W(z_1, z_2, z_3) = \left\langle \frac{1}{Z_{\phi_1} Z_{\phi_2} Z_{\phi_3}} \int \prod_{i=1}^{3} \mathrm{D}\phi_i \, \exp\left\{ -\sum_{j=1}^{3} \frac{\mathrm{i}}{2} \phi_j^{a\dagger} (z_j - H) \phi_j^a \right\} \right\rangle. \tag{125}$$

The three-point function is derived from $W(z_1, z_2, z_3)$ in the following formula:

$$G(z_1, z_2, z_3) = \lim_{n \to 0} \left( \frac{-1}{Nn} \right)^3 \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} W(z_1, z_2, z_3). \tag{126}$$

$W(z_1, z_2, z_3)$ is written in the form of the $Q$-integral

$$W(z_1, z_2, z_3) = \frac{1}{Z_Q} \int \mathrm{D}Q \, \exp\left\{ -\frac{N}{2} \mathrm{Tr}(2 \log \mathrm{i}(\boldsymbol{z} - Q) + Q^2) \right\} \tag{127}$$

where $Q$ is a $3n \times 3n$ Hermitian matrix. The most dominant saddle point $\bar{Q}$ is

$$\bar{Q} = \begin{pmatrix} w_1 I_n & 0 & 0 \\ 0 & w_2 I_n & 0 \\ 0 & 0 & w_3 I_n \end{pmatrix} \tag{128}$$

where

$$w_i \equiv \tfrac{1}{2} \left( z_i - \sqrt{{z_i}^2 - 4} \right) \qquad i = 1, 2, 3. \tag{129}$$

The dominant contributions to $W(z_1, z_2, z_3)$ becomes

$$W(z_1, z_2, z_3) = \frac{n^3}{N} F(z_1, z_2, z_3) \tag{130}$$

where

$$\begin{aligned} F(z_1, z_2, z_3) = & \frac{X_{12}}{1-X_{12}} \frac{X_{23}}{1-X_{23}} \frac{X_{31}}{1-X_{31}} + \frac{X_{31}}{1-X_{31}} \frac{X_{12}}{1-X_{12}} \frac{1}{1-X_{11}} \\ & + \frac{X_{12}}{1-X_{12}} \frac{X_{23}}{1-X_{23}} \frac{1}{1-X_{22}} + \frac{X_{23}}{1-X_{23}} \frac{X_{31}}{1-X_{31}} \frac{1}{1-X_{33}} \end{aligned} \tag{131}$$

$$X_{ij} = w_i w_j = G(z_i)G(z_j). \tag{132}$$

Thus, the three-point function in GUE is

$$G_C(z_1, z_2, z_3) = -\frac{1}{N^4} \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} F(z_1, z_2, z_3). \tag{133}$$

The connected part in this result agrees with that obtained by other methods [1, 10–12].

Finally, we calculate the connected three-point function in GSE case. That in this case is defined by

$$G(z_1, z_2, z_3) \equiv \left\langle \frac{1}{2N} \operatorname{Tr} \frac{1}{z_1 - H} \frac{1}{2N} \operatorname{Tr} \frac{1}{z_2 - H} \frac{1}{2N} \operatorname{Tr} \frac{1}{z_3 - H} \right\rangle. \tag{134}$$

The three-point function derived from the generating function is as follows

$$G(z_1, z_2, z_3) = \lim_{n \to 0} \left( \frac{-1}{2Nn} \right)^3 \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} W(z_1, z_2, z_3). \tag{135}$$

And the generating function is calculated as

$$W(z_1, z_2, z_3) = \frac{1}{Z_Q} \int \mathrm{D}Q \, \exp\left\{ -\frac{N}{2} \operatorname{Tr}(2 \log \mathrm{i}(\boldsymbol{z} - Q) + Q^2) \right\} \tag{136}$$

where $Q$ is a $6n \times 6n$ quaternion real Hermitian matrix. The most dominant saddle point $\bar{Q}$ is

$$\bar{Q} = \begin{pmatrix} w_1 I_n & 0 & 0 \\ 0 & w_2 I_n & 0 \\ 0 & 0 & w_3 I_n \end{pmatrix} \tag{137}$$

where

$$w_i \equiv \tfrac{1}{2}\left( z_i - \sqrt{z_i{}^2 - 4} \right) \qquad i = 1, 2, 3. \tag{138}$$

Performing a $1/N$-expansion for $W(z_1, z_2, z_3)$,

$$W(z_1, z_2, z_3) = \frac{4n^3}{N} F(z_1, z_2, z_3) \tag{139}$$

where

$$\begin{aligned} F(z_1, z_2, z_3) = & \frac{4X_{12}}{1-X_{12}} \frac{X_{23}}{1-X_{23}} \frac{X_{31}}{1-X_{31}} + \frac{X_{31}}{1-X_{31}} \frac{X_{12}}{1-X_{12}} \frac{1}{1-X_{11}} \\ & + \frac{X_{12}}{1-X_{12}} \frac{X_{23}}{1-X_{23}} \frac{1}{1-X_{22}} + \frac{X_{23}}{1-X_{23}} \frac{X_{31}}{1-X_{31}} \frac{1}{1-X_{33}} \end{aligned} \tag{140}$$

$$X_{ij} = w_i w_j = G(z_i)G(z_j). \tag{141}$$

The three-point function becomes

$$G_C(z_1, z_2, z_3) = -\frac{1}{2N^4} \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} F(z_1, z_2, z_3). \tag{142}$$

The connected part in this result agrees with that obtained by other methods [11, 12].

## 6. Concluding remarks

We have calculated the averaged one-point Green's functions, the wide connected two- and three-level correlators in Gaussian orthogonal, unitary and symplectic random matrix ensembles by the replica method. The one-point Green's functions have been calculated to the next-to-leading order in the $1/N$ expansion in GOE and GSE. Our results are consistent with those obtained by other methods [4, 11, 12]. We have notified that there are some regions of the spectral parameter $z$ where the employed saddle-point evaluation cannot work well for averaged Green's functions. In those regions, the fluctuation of the auxiliary variable $Q$ becomes large and the higher orders in the saddle-point expansion give the same contribution with the order $\mathrm{O}(N^0)$. To calculate the short-distance correlator we have to employ other methods for the integration instead of the saddle-point evaluation.

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